Symmetries of the ratchet current

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Recent advances in nonequilibrium statistical mechanics shed new light on the ratchet effect. The ratchet motion can thus be understood in terms of symmetry (breaking) considerations. We introduce an additional symmetry operation besides time reversal, that switches between two modes of operation. That mode reversal combined with time reversal decomposes the nonequilibrium action so as to clarify under what circumstances the ratchet current is a second order effect around equilibrium, what is the direction of the ratchet current, and what are possibly the symmetries in its fluctuations.

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I. INTRODUCTION

Irreversible thermodynamics describes the appearance of currents in macroscopic systems from specific nonequilibrium conditions. The notion of entropy production is central and makes the product of forces and fluxes. The forces are gradients of thermodynamic potentials, directly connected to differences in concentration of particles or to variations in temperature, etc. The fluxes relate to the transport of certain quantities. Basic information about the direction of these currents follows from the second law of thermodynamics (positivity of entropy production) and their response and symmetry properties are contained in the Green-Kubo and Onsager relations. Even though there is not yet a systematic nonequilibrium theory beyond first order around equilibrium, for many practical purposes that is not really problematic.

The situation is quite different and in fact, worse, for transport phenomena that arise as rectifications of fluctuations such as in Brownian motors [1,2]. We will speak here more generally about the ratchet effect. The very notion of "ratchet effect" has not been uniquely defined in the literature, perhaps witnessing the absence of a unifying understanding. Yet, a few ideas are in common. It is, e.g., emphasized that ratchets are mesoscopic systems that provide transport in spatially periodic media away from equilibrium, that ratchets are driven by fluctuations, and that the direction of transport cannot be inferred from thermodynamics [3].

In the present paper we start from the idea that symmetry breaking is central to the concept of ratchets. One is reminded of Curie's principle that "phenomena that are not ruled out by symmetries will generically happen." By symmetry, a sphere immersed in a heat bath does not move. When one makes the object asymmetric, the broken spatial symmetry no longer inhibits directed motion. However, if the heat bath is in equilibrium, the system still has unbroken time-reversal symmetry (detailed balance) which prevents motion. When finally also that time symmetry is lifted, for example, by acting with a mixture of different baths at different temperature, then the object will move. At least in principle, since on macroscopic scales the effect will in general be blurred by high inertia; the energy scales associated to the locomotion of the object have to be comparable with the thermal fluctuations induced by the surroundings.

In what follows we contribute a framework for ratchet effects, based on symmetries of the action in the path integral. Our main results are then as follows.

First, we clarify when and why the ratchet effect is second order. In a sense, to be explained, the ratchet current is then orthogonal to the entropy production. As we will specify, that harmonizes well with the understanding that "the direction of the ratchet current does not follow from the second law." Second, we make the connection with the recently studied fluctuation theorem. The ratchet work is in general the sum of three physical quantities that each satisfy a fluctuation symmetry. Sometimes, but not always, the ratchet current itself also satisfies a symmetry in its fluctuations. Finally, we discuss how to infer the direction of the ratchet current. Of course, for specific models sharper results are possibly available and the notions of ratchet work and of efficiency can sometimes be discussed in much greater detail: see, e.g., [4,5]; in [6] one considers explicitly second order currents, and fluctuations of the ratchet current have been studied in [7]. We emphasize, however, that our work concerns general methods and tools in describing the ratchet effect. From a more fundamental perspective, it illustrates and exploits the role of the time-symmetric term in the action governing the space-time histories of a system. Our analysis therefore takes part in the construction of nonequilibrium statistical mechanics beyond the linear regime.

II. RATCHET ESSENTIALS

We start by explaining our particular point of view on ratchet systems.

A. Fluctuations

Ratchet devices are best described on a microscopic or mesoscopic scale where in the usual setup one considers stochastic processes as specified from some master or kinetic equation. We do not need a specific model equation (but we will be giving examples below) and we assume that for the appropriate scale of description the distribution of histories is given after some transient time as weighted via some generalized Onsager-Machlup Lagrangian $\mathcal{L}_\lambda$,

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$$\text{Prob}[\omega] \propto e^{-\mathcal{L}_\lambda(\omega)}\mathbb{P}^0(\omega) =: \mathbb{P}^\lambda(\omega). \tag{1}$$

We explain the notation. The $\omega=(\omega_t)$ are paths or histories of the system over a certain time interval $[0,T]$, where at each time t, ω_t describes the state of the device. The weights of ω are given in terms of the functional $\mathcal{L}_\lambda$, called the action or the Lagrangian, extensive in the duration T (not explicitly indicated for simplicity of notation). All quantities derived from the Lagrangian $\mathcal{L}_\lambda$ are only defined modulo a temporal boundary term, i.e., a difference of the form $U(\omega_T)-U(\omega_0)$, and below, we often write equalities between functions of paths ω, which would be incorrect if we did not allow for such a boundary correction.

In the case of small macroscopic fluctuations, the $\mathcal{L}_\lambda$ is known as the Onsager-Machlup Lagrangian. More generally, it is simply obtained by taking a path integral representation, i.e., taking the logarithm of the path probabilities as from discrete time approximations or from so called multigate probabilities or from a Girsanov formula for Markov processes, see, e.g., [8].

The Lagrangian $\mathcal{L}_\lambda$ depends on a parameter λ which represents a particular driving that will generate the ratchet current. For $\lambda=0$, the process $\mathbb{P}^0$ is a reference process; we assume that all the nonequilibrium driving resides in $\mathcal{L}_\lambda$ so that $\mathbb{P}^0$ is in fact a corresponding equilibrium process. Nonequilibrium expectations are computed with the nonequilibrium path-space distribution (1),

$$\langle f\rangle_\lambda = \int d\mathbb{P}^0(\omega) f(\omega) e^{-\mathcal{L}_\lambda(\omega)}$$

for the normalized expectation of a function $f(\omega)$ in histories ω.

B. Modes of operation

A ratchet device can be considered as a motor that has various pathways or channels to complete its working cycle, available at different times or at different locations. No explicit thermodynamic force needs to be specified. In general, the state of the ratchet is represented by two coordinates: x, a one-dimensional cyclic coordinate which gives the position of the motor, and k, mostly discrete and which specifies additional information concerning the specific channel or mode of operation. The coordinate k can be spatial (e.g., like in Feynman's ratchet and pawl), it can determine the type of environment (when the motor interacts with a gas consisting of multiple species which are not in equilibrium with each other), it can specify the potential (like in a flashing ratchet) or the value of some time-dependent external field. In some cases, the different modes of operation could represent different energy levels of the system and the switching then results from contact with a heat bath, cf. thermoelectric effects as in [9]. The possible values of k can mostly be associated to the action of different reservoirs, spatially or temporally. In the present paper we restrict ourselves to two values for k and hence in particular no treatment will be given involving a continuum of different heat baths such as, for example, in the Büttiker ratchet realizing a space-dependent temperature [10]. In summary, the paths ω we have in mind when writing Eq. (1) also include the information of what temperature, or what potential, etc., is used (k coordinate) at what time, and not only the position of the motor itself (x coordinate).

Since k takes two values, these channels can be divided in a set of pairs, and we can usefully define a transformation between the two members of the pair. To be specific, imagine a Markov jump process for which the paths ω correspond to sequences x_j,k_j of positions and modes, respectively, and of jump times t_j:

$$\omega = (x_1,k_1,t_1;x_2,k_2,t_2;\ \dots\ ;x_n,k_n). \tag{2}$$

The k_j take the values 1, 2. We can now introduce a transformed path

$$\Gamma\omega = (x_1,\bar{k}_1,t_1;x_2,\bar{k}_2,t_2;\ \dots\ ;x_n,\bar{k}_n),$$

where $\bar{k}_j=3-k_j$ switches $1\leftrightarrows 2$ for the mode of operation. More generally and for each path ω we can associate to it a transformed path $\Gamma\omega$, obtained by switching k's in each step of the path and thus switching the modes of operation of the motor without touching the trajectory of the particle itself. We emphasize that the symmetry Γ, called mode reversal, acts directly on path space as we include in the history the setting of the driving or of the environment. E.g., Γ allows us to exchange two different potentials or temperatures, etc. One can have in mind that Γ is (effectively) a sign reversal of the thermodynamic forces, e.g., $\lambda\to-\lambda$. In the case of devices with external periodical forcing, Γ corresponds to shifting each path by one-half of the period of the external force.

Besides mode reversal and as essential in all nonequilibrium systems one can also apply time reversal. One then compares the weight of a trajectory ω with that of its time-reversal $\theta\omega$: $(\theta\omega)_t=\omega_{T-t}$. We restrict us to variables like particle positions, and we do not consider here variables that are odd under kinematical time reversal (like velocities). The difference between the probabilities for ω and $\theta\omega$ measures the irreversibility, as has been expressed in a number of fluctuation relations over the last years, see [11] for a review.

It is the breaking of the Γ symmetry, combined with the breaking of detailed balance, that generates the nonequilibrium ratchet effect. It generates a nonzero ratchet current J_r measuring the cycling speed, at least when there are no further symmetries that would forbid $J_r\neq 0$. We now consider the symmetry properties of the path-dependent ratchet current J_r. In contrast with many situations close to equilibrium, we need to introduce yet other considerations than strictly related to entropy production or time reversal (breaking). Now comes the relevance of the symmetry operation Γ. We say that J_r is a *ratchet current* (associated to the operation Γ) if it satisfies both

$$J_r(\Gamma\omega) = J_r(\omega),$$

$$J_r(\theta\omega) = -J_r(\omega). \tag{3}$$

The first symmetry of J_r under Γ means that the ratchet current simply counts the number of completed cycles (in the x coordinate) no matter along what channel (choices of k coordinate) it was taken; as a current counting the steps of the ratchet in ω we naturally ask that $J_r(\omega)$ is antisymmetric under time reversal θ.

III. FIRST ORDER VS SECOND ORDER

We require that the equilibrium situation is θ symmetric,

$$\mathbb{P}^0(\theta\omega) = \mathbb{P}^0(\omega) \tag{4}$$

which implies that in equilibrium $\langle J\rangle_0=0$ for all time-antisymmetric observables J. The nonequilibrium driving breaks the time symmetry and we let $S_\lambda=S$ be the θ-antisymmetric part of the Lagrangian, i.e.,

$$S = \mathcal{L}_\lambda(\theta\omega) - \mathcal{L}_\lambda(\omega). \tag{5}$$

It turns out that the variable S can be identified with the path-dependent entropy production appropriate to the scale of description [11,12], always up to a total time difference. Obviously, $S(\theta\omega)=-S(\omega)$.

A. Orthogonality

For ratchets it is very useful to employ also the mode reversal Γ, and to put ω in the balance vs $\Gamma\omega$. To start we also ask here that

$$S(\Gamma\omega) = -S(\omega) \tag{6}$$

which is straightforward in most concrete models (think, e.g., of heat conduction where one exchanges the temperatures of baths for a fixed history ω). Remark that the entropy production S and the ratchet current J_r then behave differently under the symmetry Γ, but identically under the symmetry θ.

Clearly, from the properties $S\Gamma=-S$, $J_r\Gamma=J_r$ follows that the mutual covariance between S and J_r equals zero,

$$\int Q(d\omega)J_r(\omega)S(\omega) = 0, \quad \int Q(d\omega)S(\omega) = 0, \tag{7}$$

for no matter what Γ-invariant distribution Q. The identity (7) expresses an orthogonality or independence between the variable entropy production and the ratchet current. It announces that the ratchet effect plays beyond irreversible thermodynamics and there arises, for example, the problem of determining the direction of the ratchet current.

One can indeed learn something about the ratchet effect by the usual perturbation theory around equilibrium. One then expands the nonequilibrium state $e^{-\mathcal{L}_\lambda}\mathbb{P}^0$ around equilibrium $\mathbb{P}^0$ to obtain, via Eqs. (4) and (5),

$$\langle J_r\rangle_\lambda = \frac{1}{2}\langle J_r S_\lambda\rangle_0 + O(\lambda^2). \tag{8}$$

The consequence of Eq. (7) now appears. In many cases, including almost all flashing ratchets, the equilibrium process is invariant under Γ. Then we can take $Q(\omega)=\mathbb{P}^0(\omega)$ in Eq. (7) and $\langle J_r S_\lambda\rangle_0=0$. As a result, from Eq. (8) we see that the ratchet current vanishes in first order in λ. The reason is the invariance of the equilibrium process under Γ combined with the antisymmetry of the entropy production S under Γ. That appears to be the general mechanism when obtaining ratchet effects only in second order around equilibrium. At the same time, we see that first order ratchets appear when the equilibrium state $\mathbb{P}^0$ is not Γ invariant; see [13] for a simple example.

B. Ratchets with load

When one attaches a load to extract work from the ratchet effect, the above description must be modified. Applying a load is effectively coupling the ratchet current to the entropy production. It is now no longer true that the entropy production S is antisymmetric under Γ and the relation (6) no longer holds. To further resolve the (anti)symmetries, we decompose S_λ into

$$S_\lambda = S_\lambda^+ + S_\lambda^-,$$

where $S_\lambda^+=S_\lambda^+\Gamma$ ($S_\lambda^-=-S_\lambda^-\Gamma$) is (anti)symmetric under Γ. As an example, we can already think of a heat engine working between inverse temperatures β_1 and β_2. The variable entropy current is $S=\beta_1J_1+\beta_2J_2$ where J_i is the heat current into reservoir i, while the delivered work equals $-W=J_1+J_2$ (energy conservation). Then,

$$S = \frac{1}{2}(\beta_1-\beta_2)(J_1-J_2) + \frac{1}{2}(\beta_1+\beta_2)W. \tag{9}$$

We think of the exchange of heat baths as a mode reversal and we can take $\lambda\sim\beta_1-\beta_2$. The first term in Eq. (9) is antisymmetric under the exchange $\beta_1\leftrightarrow\beta_2$ and the second term (containing the work W) is symmetric under Γ. Quite generally, the term S^+ turns out to be proportional to the work done on the ratchet, as a function on path space. Assuming that ratchet work is proportional to the number of completed cycles (as can be checked quite often) we write the work as $S^+=-fJ_r$ for a constant load f. As a consequence, the linear term in Eq. (8) gets rewritten as

$$\langle J_r\rangle_{\lambda,f} = \frac{1}{2}\langle J_r S_\lambda^-\rangle_0 - \frac{1}{2}f\langle J_r J_r\rangle_0 + O(\lambda^2,f^2).$$

Again, the first term on the right (coupling heat dissipation with the ratchet current) vanishes if the equilibrium state $\mathbb{P}^0$ is Γ invariant and the response of the ratchet current to the load is in first order determined by a current-current autocorrelation (the second term on the right).

IV. EXAMPLES

Ratchets allow motion without the application of net thermodynamic forces. The difference between a ratchet and a *perpetuum mobile of the second kind* arises from the nonequilibrium condition. Depending on the specific nature of the nonequilibrium one distinguishes different kinds of ratchets. As a result the above notions are realized in a somewhat different way for different ratchets (flashing ratchets, rocked ratchets, Feynman ratchets, etc.). Yet our presentation (and examples below) are restricted to bimodal operations only (two possible values of k) excluding therefore, e.g., Büttiker-

Landauer ratchets [10]. To fix the ideas and to illustrate the basic concepts, we consider here two classes of ratchet systems.

A. Two-temperature ratchet

A particle travels on a periodic landscape, modeled by a double ring whose sites are indexed by (x,k) with $x=0,\ldots,L$ and $k=1,2$. Site 0 is identified with L. An asymmetric potential function $V(x)$ is given. In each step the particle can either jump from (x,k) to $(x\pm1,k)$, or it can change its k coordinate while keeping x unchanged. One could have in mind that the particle moves on the interface between two gas reservoirs; whenever $k=1$, it interacts with reservoir 1 and analogously for $k=2$. The reservoirs have respective inverse temperatures $\beta_{1,2}$. The dynamics is given by a Markov jump process with jump rates

$$c((x,k),(y,k)) = g_k(x,y)e^{-\beta_k[V(y)-V(x)]/2} \tag{10}$$

for jumps from x to a nearest neighbor $y=x\pm1$ on the ring, and

$$c((x,k),(x,k')) = c((x,k'),(x,k)) = h(x) \tag{11}$$

for a change of $k\to k'$. In going from (x,k) to (y,k), the particle absorbs energy $V(y)-V(x)$ from reservoir k. We demand that $g_k(x,y)=g_k(y,x)$ and the symmetry (11) to assure that the only source of entropy creation in the jump is by the transfer of heat $V(y)-V(x)$ (see also the first paragraph of Sec. III). The functions $g_k(x,y)$ can, for example, include details about the chemical potential of the reservoir, or more generally, about the contact between the reservoir and the particle. We remark that an eventual chemical potential does not cause any entropy production since no gas particles are being transported between the two reservoirs.

The driving λ can then be identified with the difference between the two reservoirs, say in terms of $\beta_1-\beta_2$ and $g_1(x,y)-g_2(x,y)$. We hence make the assumption that $g_1(x,y)=g_2(x,y)$ when $\beta_1=\beta_2$, corresponding to equilibrium. The paths ω correspond to sequences of positions x_j,k_j and of jump times t_j:

$$\omega = (x_1,k_1,t_1;x_2,k_2,t_2;\ldots;x_n,k_n) \tag{12}$$

as in Eq. (12). Time reversal θ (for some large T) transforms the path ω into $\theta\omega=(x_n,k_n,T-t_{n-1},k_{n-1};\ldots;x_2,k_2,T-t_1;x_1,k_1)$. The mode reversal Γ exchanges the reservoirs and it works on the k_j's exchanging $k=1,2$ The two reservoirs are identical in the equilibrium process $[\lambda=0\Rightarrow\beta_1=\beta_2,g_1(x,y)=g_2(x,y)]$.

The antisymmetric term (5) under time reversal in the Lagrangian can be obtained from computing

$$\begin{aligned} S(\omega) &= \ln\frac{\mathbb{P}(\omega)}{\mathbb{P}(\theta\omega)} \\ &= \ln\frac{c((x_1,k_1),(x_2,k_2))\cdots c((x_{n-1},k_{n-1}),(x_n,k_n))}{c((x_n,k_n),(x_{n-1},k_{n-1}))\cdots c((x_2,k_2),(x_1,k_1))} \end{aligned}$$

or

$$S(\omega) = \sum_{j=1}^{n-1}\beta_{k_j}[V(x_{j+1})-V(x_j)] \tag{13}$$

which is the sum of changes in the entropy of the gases [note that the jumps where the k coordinate changes do not enter $S(\omega)$]. The particle itself is thought of as microscopic and not contributing to the entropy, so that Eq. (13) is the path-dependent entropy production.

Clearly, S is antisymmetric under time reversal. There is another way of writing Eq. (13) to make clear that S is also antisymmetric under Γ:

$$\begin{aligned} S(\omega) &= -\beta_1\sum_{j:k_j=1}[V(x_{j+1})-V(x_j)] - \beta_2\sum_{j:k_j=2}[V(x_{j+1})-V(x_j)] \\ &= -(\beta_1-\beta_2)\sum_{j:k_j=1}[V(x_{j+1})-V(x_j)] - \beta_2[V(x_n)-V(x_1)] \\ &= -(\beta_1-\beta_2)\sum_{j:k_j=1}[V(x_{j+1})-V(x_j)]. \end{aligned} \tag{14}$$

The last equality illustrates our convention that all path-dependent quantities are written modulo a total time difference.

Clearly, the ratchet current $J_r(\omega)$ is a function of $\tilde{\omega}=(x_1,t_1;\ldots;x_n)$ only and it does not depend on the k_j's. Its mean $\langle J_r\rangle$ is generically nonzero when V is asymmetric (and no other accidental symmetries are present). The ratchet is second order [this is due to our assumption that $g_1(x,y)=g_2(x,y)$ when $\beta_1=\beta_2$]; the entropy production (13) is not of the form $\mathcal{F}J_r$.

B. Flashing ratchet

In the previous example, it was the environment (and specifically the temperature) that was effectively changing between two possible values. We can also take the time dependence in the shape of the potential. As another difference we consider now a Langevin setup. Again it concerns a second order ratchet.

Consider a particle in a spatially periodic landscape with the potential flashing between two potential functions V_{+1} and V_{-1}, both periodic functions $V_{\pm1}(x)=V_{\pm1}(x+L)$. Again, one has to eliminate additional symmetries, like mirror symmetry of the potentials or supersymmetry [2], to get a nonzero ratchet current.

The nonequilibrium parameter λ measures the difference between the two potentials parametrized as $V_{\pm1}=V\pm\lambda W$. The particle is in contact with a heat bath at inverse temperature β. We model its motion by the overdamped Langevin equation

$$\dot{x}_t = -V'_{k(t)}(x_t) + \xi_t, \tag{15}$$

where ξ_t is a fluctuating Gaussian force with white noise statistics: $\langle\xi_t\rangle=0$ and $\langle\xi_s\xi_t\rangle=2\beta^{-1}\delta(t-s)$. The time dependence $k_t=\pm1$ is arbitrary. The reference process has $\lambda=0$, meaning that the potential is fixed equal to V. Under Itô convention, one shows

$$\mathcal{L}_\lambda = \frac{\beta}{2}\left(\lambda \int dx_t k_t W'(x_t) + \lambda \int dt k_t V'(x_t) W'(x_t) + \frac{\lambda^2}{2}\int dt W'^2(x_t)\right). \quad (16)$$

The paths are given as $\omega_t=(x_t,k_t)$ with time reversal implemented by (for some large T) $\theta(x_t,k_t)=(x_{T-t},k_{T-t})$ and

$$S = \mathcal{L}_\lambda \theta - \mathcal{L}_\lambda = -\beta\lambda \int dt \dot{k}_t W(x_t) \quad (17)$$

which is β times the dissipated power through the external forcing. The mode reversal Γ switches potentials, $\Gamma(x_t,k_t)=(x_t,-k_t)$ and one observes that $S\Gamma=-S$.

V. RATCHET FLUCTUATIONS

The two symmetry operations Γ and θ suggest a natural decomposition of the Lagrangian $\mathcal{L}_\lambda$. From now on we assume that $\theta\Gamma=\Gamma\theta$ (commutativity [23]). We write

$$R = R_\lambda = (\mathcal{L}_\lambda \theta\Gamma + \mathcal{L}_\lambda \Gamma - \mathcal{L}_\lambda \theta - \mathcal{L}_\lambda)/2 \quad (18)$$

for the part that is antisymmetric under Γ and is symmetric under θ. The Lagrangian has the form

$$\mathcal{L}_\lambda = \mathcal{L}_\lambda^+ - \frac{1}{2}[R_\lambda + S_\lambda], \quad (19)$$

where $\mathcal{L}_\lambda^+$ is (θ,Γ) invariant.

One can now verify that ratchet models typically satisfy various fluctuation theorems. In brief, when $\mathbb{P}^0$ is (θ,Γ) invariant, then for all the three choices $V=S, R+S^+, R+S^-$,

$$\frac{\mathbb{P}^\lambda(V=v)}{\mathbb{P}^\lambda(V=-v)} = e^v. \quad (20)$$

For $V=S$, Eq. (20) is similar to the Gallavotti-Cohen fluctuation symmetry for the fluctuations of the entropy production [14,15]; for $V=R+S^+$, Eq. (20) has been derived in [16]; finally, Eq. (20) also holds for $V=R+S^-$. The reason why in all these cases one finds that fluctuation relation is that S, $R+S^-$, and $R+S^+$ are the antisymmetric parts in the Lagrangian $\mathcal{L}_\lambda$ under respectively the symmetries θ,Γ and $\theta\Gamma$. The relation (20) can in each of the three cases be directly verified from computing the ratio $\mathbb{P}^\lambda(\omega)/\mathbb{P}^\lambda(Y\omega)$ for transformations $Y=\theta,\Gamma,\theta\Gamma$ in Eq. (1), and from combining that with the decomposition (19). In order to control temporal boundary terms, it is assumed that the system itself has a bounded state space; otherwise, some extended fluctuation symmetry can be expected, see [17,18].

Observe also that the ratchet work $S^+=[S+(R+S^+)-(R+S^-)]/2$ is a sum of three observables, each of which satisfies a fluctuation theorem (20).

A natural question is whether the ratchet current J_r itself satisfies a fluctuation symmetry. In general, the answer seems to be negative, but nevertheless it is possible to construct classes of ratchets where that symmetry is verified, as is also remarked for some specific models in [19,20], and as now will be shown.

We come back to the two-temperature ratchet of Sec. IV A. We consider the limiting case of a very rapid changing of the reservoir (k coordinate), hence the limit $h(x)\uparrow+\infty$ in Eq. (11). In that limit, we obtain effectively a mixture of the two reservoirs. Another possible realization is obtained by thinking of the particle as a rigid body extended and connected at its ends to two different reservoirs. Then, we have a simple model of the Feynman-Smoluchowski ratchet much in the spirit of [21] but in the overdamped limit. The two modes of operation still correspond to the two reservoirs but with respect to Eq. (12), we make now a more coarse grained description: we only look at the particle jumps (forgetting about which reservoir caused it), i.e., the jump rates are now between x and y and they are given by the sum $c(x,y)=c((x,1),(y,1))+c((x,2),(y,2))$. In other words, we collect several of the original paths ω of Eq. (12) into one and the same new path $\tilde{\omega}=(x_1,t_1;x_2,t_2;\ldots;x_n)$. Obviously now the Γ symmetry has left the stage and there is effectively only one possible channel (though of course, if one wants to keep track of the physical entropy production, one still has to distinguish which reservoir "caused" what transition). The corresponding path space distribution is

$$\tilde{\mathbb{P}}(\tilde{\omega}) = \sum_{\omega\to\tilde{\omega}} \mathbb{P}^\lambda(\omega), \quad \tilde{\mathbb{P}}(\tilde{\omega}) \propto e^{-\tilde{\mathcal{L}}(\tilde{\omega})} \quad (21)$$

with a new Lagrangian $\tilde{\mathcal{L}}$. The key observation is that pathwise, its antisymmetric component $\tilde{\mathcal{L}}(\theta\tilde{\omega})-\tilde{\mathcal{L}}(\tilde{\omega})$ is proportional to the ratchet current

$$\tilde{\mathcal{L}}\theta - \tilde{\mathcal{L}} = aJ_r \quad (22)$$

with a constant a that can be computed explicitly. More specifically, from Eq. (10) we have arrived at a Markov process with rates

$$c(x,y) = \sum_{k=1}^{2} g_k e^{-\beta_k[V(y)-V(x)]/2},$$

where we take the prefactors g_k independent of the position. To compute Eq. (22) we must make the ratio of the probabilities of $\tilde{\omega}$ and $\theta\tilde{\omega}$ and we concentrate on the jump times in which the particle moves either forward or backward with one step. The ratchet current $J_r(\tilde{\omega})$ is the net number of steps forward in time span T and is a sum over all the jump times of $+1$, respectively -1, as the jump carries forward, respectively backward. Obviously, each time the particle has completed one cycle, say, in the positive direction, each bond has carried exactly one net jump forward. Therefore Eq. (22) is

$$\ln \frac{\tilde{\mathbb{P}}(\tilde{\omega})}{\tilde{\mathbb{P}}(\theta\tilde{\omega})} = aJ_r(\tilde{\omega}) + o(T)$$

for

$$a = \frac{1}{L+1}\sum_{x=0}^{L} \ln \frac{\sum_{k=1}^{2} g_k e^{-\beta_k[V(x+1)-V(x)]/2}}{\sum_{k=1}^{2} g_k e^{-\beta_k[V(x)-V(x+1)]/2}}.$$

By standard arguments it now follows that for $j\sim T$

$$\frac{\mathbb{P}^\lambda(J_r=j)}{\mathbb{P}^\lambda(J_r=-j)}=e^{aj} \tag{23}$$

as $T\uparrow+\infty$, which is a fluctuation symmetry for the ratchet current. In particular $a\langle J_r\rangle_\lambda \geq 0$, which obviously determines the sign of the ratchet current.

Note that in this limit, there is a new accidental symmetry possible; if $g_1(x,y)=g_2(x,y)$ for some nonzero λ, then one easily checks that $c(x,y)/c(y,x)=e^{A(y)-A(x)}$ for some function A. This "effective" detailed balance condition immediately implies $J_r=0$. The same kind of symmetry can be seen in the ratchet [21] if one models the contact with the thermal baths by Langevin forces (instead of a Boltzmann equation, as in done in [21]).

VI. DIRECTION OF RATCHET CURRENTS

In first-order ratchets, one can interpret Eq. (8) as a principle for determining the direction of the ratchet current close to equilibrium, providing a simple mathematical explanation of the ideas in [22]. Indeed, since $\mathbb{P}^0[J_r>0]=\mathbb{P}^0[J_r<0]$, we can evenly split

$$\langle J_r S_\lambda\rangle_0 = 1/2\langle J_r S_\lambda | J_r>0\rangle_0 + 1/2\langle J_r S_\lambda | J_r<0\rangle_0. \tag{24}$$

Combine that with the fact that $\langle S_\lambda\rangle_0=0$ to conclude that if the entropy production S_λ is overwhelmingly positive in one of the two subensembles $J_r>0$ or $J_r<0$, then the ratchet current has the sign as in that subensemble.

For more general ratchets, one can use the consequences of the fluctuation theorems (20). It implies that S, $R+S^+$, and $R+S^-$ are all positive with a probability that exponentially approaches 1 as the duration $T\uparrow\infty$. In principle, that determines the direction of the ratchet current.

To be more specific, we consider unloaded ratchets for which the first order around equilibrium vanishes, see the discussion around Eq. (8). Then, the first nonvanishing order is given by

$$\langle J_r\rangle_\lambda = \frac{1}{4}\langle J_r S_\lambda R_\lambda\rangle_0 + O(\lambda^3). \tag{25}$$

Hence one has to study the sign of $S_\lambda R_\lambda$ in the two equilibrium subensembles $J_r>0$ and $J_r<0$. Typical trajectories are characterized by having positive entropy production $S_\lambda>0$. Yet, that does not yet fix the direction of the ratchet current in the case of second order. The time-symmetric term R_λ must, however, also be positive for typical paths. That selects within the class of paths where $S_\lambda>0$ what the direction of the current will be.

VII. CONCLUSIONS

Fluctuations are driving the ratchet effect. It is therefore important to investigate the structure of the action in the path integral governing the path probabilities. Another symmetry transformation Γ (mode reversal) appears that together with time reversal decomposes the nonequilibrium action. The term in the Lagrangian action that is symmetric under time reversal but is antisymmetric under mode reversal contributes significantly to determining the direction and the nature of the fluctuations of the ratchet current. That effect is most outspoken for second order ratchets.

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[23] That is generally true if the ratchet coordinate can be separated as (x,k), Γ acts by changing k, and θ does not mix x and k. Both examples in Sec. IV have that property.